PIECEWISE LINEAR SURVIVAL FUNCTION: A CONTINUOUS CONTEXT STUDY

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Abstract. We present here some preliminary results of an ongoing study. The subject of this work is a continuous survival function composed of two distinct, linear segments.

1. Introduction

Typically, the survival function, that we will note $S(x)$[1], when plotted on the y-axis depicts the gradual extinction of a generation. Individuals will gradually die off until ω, the age reached by the longest-lived one. $S(x)$ is always non-negative and never increases: at best, no one dies—at least for some time, because at age ω the last survivor passes away.

Specifically, we will examine the case where the function consists of two identifiable, but joined, linear segments, along with some characteristics of the associated stationary population; hereafter, we will not repeat the distinction between the function itself and the associated stationary population.

Let us start with the simplest of these functions, the linear one (or De Moivre):

$$S^{DM}(x) := S(0)\left(1-\frac{x}{\omega}\right) = S(0)\,\frac{\omega-x}{\omega} \qquad (1)$$

We split it at an age K not exceeding the maximum longevity ω: in this way S becomes a piecewise continuous linear function. Here we distinguish two parts:

- The first segment S_1, from birth to K, will have a negative slope.
- The second segment S_2, from K to ω, will be similar but with a different inclination; this time it is constrained, since the function must reach zero at the final age ω. Therefore, it will correspond to a linear function starting not from age zero, but from age K. As will be shown, despite a more complex premise, the outcomes are here simpler than those of the first segment.

[1] $l(x)$ is also used in the literature.

The respective formulations are characterized by their linearity (technically, a constant first derivative), differing in points of origin and slopes respectively characterized by the non-negative parameters λ and ν:

$$S(x) = \begin{cases} S_1(x) = S(0)\left(1 - \frac{\lambda}{\omega} x\right) & x \in [0, K) \\ S_2(x) = S(K)\left[1 - \frac{\nu}{\omega}(x - K)\right] & x \in [K, \omega) \end{cases} \quad (2)$$

Since S is positive and $0 \leq K \leq \omega$, we have the constraint:

$$\lambda \in \left[0, \frac{\omega}{K}\right] \quad (3)$$

From $\lambda \leq \frac{\omega}{K}$ follows:

$$\begin{cases} K < \omega & \lambda \leq 1 \\ K \leq \frac{\omega}{\lambda} & \lambda > 1 \end{cases} \quad (4)$$

Since the S is continuous, we need to set:[2]

$$S_1(K) = S_2(K) = S(0)\frac{(\omega - \lambda K)}{\omega} \quad (5)$$

Moreover, the function S must be vanished when $x = \omega$. This implies that:

$$S_2(\omega) = 0 \iff \nu = \frac{\omega}{\omega - K} \quad (6)$$

Equations (5) and (6) allows us to rewrite the function S without explicating parameter ν:

$$S(x) = \begin{cases} S(0)\left(1 - \frac{\lambda}{\omega} x\right) & x \in [0, K) \\ S(0)\left(\frac{\omega - \lambda K}{\omega}\right)\left(\frac{\omega - x}{\omega - K}\right) & x \in [K, \omega) \end{cases} \quad (7)$$

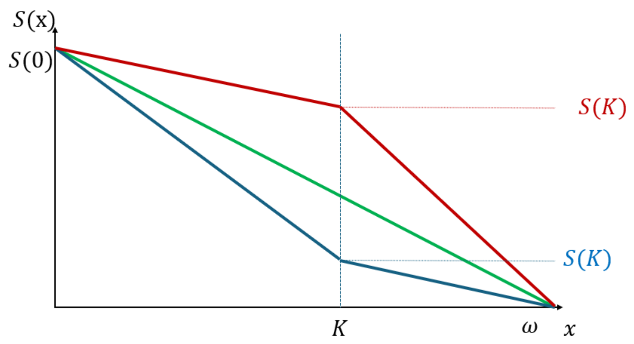
[2] Equation (5) will also have to hold for other biometric variables; incidentally, it is also a method of verifying the correctness of calculations.

We have avoided a more in-depth discussion of the derivation of the model due to space constraints.

The slopes of the two segments are closely related: a relationship conditioned by the point at which the function bends. The earlier the value of K, the less abrupt the correction needed to bring the curve to the bottom right corner: $S(\omega)$ must, in fact, have coordinates (0, ω).

Below is shown how our survival function might appear in the three main cases, which we are overlaying here—red for low mortality ($K < 1$), green for the linear case $K = 1$), and blue for high mortality ($K > 1$):

Figure 1 – *The three cases of the piecewise continuous linear survivor function S(x).*



2. Biometric functions

2.1 Deaths

In the continuous analysis, density of deaths is by definition equivalent to the absolute value of the slope of the survivors; in our case:

$$d(x) = \begin{cases} \lambda \frac{S(0)}{\omega} & x \in [0, K) \\ \frac{\omega - \lambda K}{\omega - K} \frac{S(0)}{\omega} & x \in [K, \omega) \end{cases} \tag{8}$$

Deaths are therefore constant with respect to age, as long as one remains within the same segment of the function: this is why we might omit x.

Constant yearly deceases imply that the initial cohort $S(0)$ is equal to the simple product of this annual deaths and the maximum age reached by the generation: such principle, valid immediately for De Moivre, must in our case be adapted to the existence of two segments with annual deaths respectively d_1 and d_2:

$$S(0) = Kd_1 + (\omega - K)d_2 = \frac{S(0)}{\omega}[\lambda K + (\omega - \lambda K)] \tag{9}$$

We immediately notice from (8) that the slope of the first segment is directly proportional to λ, while that of the second segment to $\frac{\omega-\lambda K}{\omega-K}$ moves in the opposite direction with respect to λ. If the first segment is steeper than the linear case (where λ = 1), then the slope of the second segment will have to compensate with a softer decline in order to reach the bottom right vertex, corresponding to extinction at age ω; and vice versa (see Figure 1).

Now we may introduce the instantaneous mortality rate:

$$\mu(x) = \begin{cases} \frac{\lambda}{\omega-\lambda x} & x \in [0, K) \\ \frac{1}{\omega-x} & x \in [K, \omega) \end{cases} \tag{10}$$

Observe that in the right-piece of the function μ, the parameter λ does not influence the mortality, which is the same as the De Moivre.

2.2 *Resistance*

Petrioli (1982, pp.177-178) introduces some physical analogies in the field of demography, comparing the number of deaths in an age interval to the "work" of mortality; he then continues with its "average power" π, which is nothing more than the average of deaths per unit of time: [3]

$$\pi(x + \Delta x) \coloneqq \frac{M(x+\Delta x)}{\Delta x} = \frac{S(x)-S(x+\Delta x)}{\Delta x} \tag{11}$$

Finally, he introduces the "resistance function" *r*, which "depends on the law of elimination and is particularly sensitive to its variations" (p.178):

$$r(x) \coloneqq \frac{\pi(x,\omega)}{\pi(0,x)} = \frac{xS(x)}{(\omega-x)[S(0)-S(x)]} \tag{12}$$

By applying the general formula (12) to our case, we get:

[3] $M(x, x + \Delta x) = S(x) - S(x + \Delta x)$ holds true in every survival function.

$$r(x) = \begin{cases} \frac{\omega-\lambda x}{\lambda(\omega-x)} & x \in [0,K) \\ \frac{x(\omega-\lambda K)}{\lambda K(\omega-x)+\omega(x-K)} & x \in [K,\omega) \end{cases} \tag{13}$$

Notice that at $x = K$ it holds in both cases:

$$r(K) = \frac{\omega-\lambda K}{\lambda(\omega-K)} \tag{14}$$

At this point, we just need to examine the behavior of $r(x)$, also to check if in our case the "bell-shaped trend" emerges as described in Petrioli and Berti 1979 p. 20: this aspect is currently under investigation.

2.3 Life expectancy

The first step is to retrocumulate the years lived. We start with T_2, because in the logic of retrocumulation T_1 presupposes it:

$$T_2(x) = \int_{x\geq K}^{\omega} S_2(\xi)d\xi = \frac{S(K)}{\omega-K}\int_{x\geq K}^{\omega}(\omega-\xi)\, d\xi = \frac{S(0)}{\omega}\frac{\omega-\lambda K}{\omega-K}\frac{(\omega-x)^2}{2} \tag{15}$$

where ξ is the mute variable of integration. The subscript notation related to the segment is omitted in $S(K)$ because the value coincides in both segments; we will continue to do so where possible. We naturally have also: $T_1(K) = T_2(K) = T(K)$.
Hence:

$$T_1(x) = \int_x^K S_1(\xi)d\xi + T(K) = \frac{S(0)}{2\omega}\left[(\omega^2 - 2\omega x + \lambda x^2) + (1-\lambda)K\omega\right] \tag{16}$$

In particular, it holds:

$$T_1(0) \coloneqq T(0) = S(0)\frac{\omega+(1-\lambda)K}{2} \tag{17}$$

From T, we finally obtain the life expectancy; by applying the general formula $e(x) \coloneqq T(x)/S(x)$, we have:

$$e(x) = \begin{cases} e_1(x) = \frac{(\omega^2-2\omega x+\lambda x^2)+(1-\lambda)K\omega}{2(\omega-\lambda x)} & x < K \\ e_2(x) = \frac{\omega-x}{2} & x \geq K \end{cases} \tag{18}$$

e_1 and e_2 being the left and right sides of the function e respectively, we could alternatively write the former by separating the elements involving λ:

$$e_1(x) = \frac{(\omega^2-2\omega x+K\omega)-\lambda(K\omega-x^2)}{2(\omega-\lambda x)} \tag{19}$$

The value of the second segment e_2 is no longer influenced by λ and K, and it is equivalent to that of De Moivre. In the first segment, however, both ones appear. In particular, the parameters in the second segment affect the number of survivors, but not their life expectancy because they cancel out with $T_2(x)$.

It is unnecessary to add that $e_1(K) = e_2(K)$,.

At birth, we have:

$$e_1(0) := e(0) = \frac{\omega+(1-\lambda)K}{2} = e^{DM}(0) + \frac{(1-\lambda)K}{2} \tag{20}$$

We derive $e_1(x)$ to systematically discover the effect that age, the slope of the first segment, and the bending point have on this:

$$\frac{\partial e_1(x)}{\partial x} = -\frac{\lambda^2x^2-2\lambda\omega x+[(2-\lambda)\omega-\lambda(1-\lambda)K]\omega}{2(\omega-\lambda x)^2} \tag{21}$$

$$\frac{\partial e_1(x)}{\partial \lambda} = -\frac{\omega(K-x)(\omega-x)}{2(\omega-\lambda x)^2} \tag{22}$$

$$\frac{\partial e_1(x)}{\partial K} = \frac{(1-\lambda)\omega}{2(\omega-\lambda x)} \tag{23}$$

Note that everything depends on the numerator—naturally considering any preceding "-" —as the denominator is never negative.

Certainly negative is (22), and it is intuitive why: as λ increases, the area under the survival curve in the ages under K decreases (the critical value ω/K coincides with the slope that zeroes out survivors at the breaking point.); once lost some of such initial life-space because of a steeper λ, the tied behaviour of S_2 does not allow for a full catching-up.

The sign of (23) clearly depends on λ: above the value of 1 it is positive, negative for values below; indeed, increasing K extends the action of λ for a longer period before the final compensatory decline.

The first derivative is the most difficult to evaluate and keeps holds a surprise: it is not always negative, as one might expect: Section 3 delves into this further.

3. Does aging harm health?

Here we study the trend of life expectancy by age in the first segment of the function: after *K*, we are indeed in a straightforward variation (albeit delayed) of De Moivre's law. In this paragraph, for the sake of clarity and ease of treatment, we also exclude the degenerate case *K* = 0, which corresponds to the simple .

As a general principle, *e*(*x*) typically declines because, as intuition suggests, over time individuals consume their available life. The inevitability of death implies life expectancy generally decreases over the course of existence.

However, can this function also rise at certain somewhere? The answer, well-known to demographers, is: yes. This typically occurred at the beginning of life, when high mortality rates now were countered by a perspective of lower risks in the future. This is a characteristic of survival curves in populations characterized by very high infant mortality, typically those in pre-transitional stages: hence the use of the imperfect tense. Almost certainly, even today we would observe reversals in life expectancy trends if we could measure it from the moment of conception.

There exists a general formulation for the derivative of life expectancy with respect to age, following directly from the application of Leibniz Rule and making it more intuitive how life expectancy can rise in certain phases; specifically, those characterized by strong immediate mortality followed by high survival for the remainder of life:

$$\frac{de(x)}{dx} = \mu(x)e(x) - 1 \tag{24}$$

A stark example: if one person with a life expectancy of 60 years were forced to play the unhealthy game of Russian roulette, his life expectancy would drop to 50 years, returning then immediately to 60 after (potentially) surviving the ordeal; in this calculation, we neglect the time taken for the unpleasant test (de minimis non curat prætor) and use a six-chamber revolver. We apply here the general formula $e(x) = e_\alpha p_\alpha + e_{\tilde{\alpha}}(1 - p_\alpha)$, more statistical than demographic, where p is the survival probability and $\tilde{\alpha}$ denotes cases other than α.

Now we may focus on equation (21) considering its numerator as a quadratic in the variable x:

$$\frac{\partial e_1(x)}{\partial x} = -\frac{\lambda^2x^2 - 2\lambda\omega x + [(2-\lambda)\omega - \lambda(1-\lambda)K]\omega}{2(\omega - \lambda x)^2} =: -\frac{\Xi(x)}{2(\omega - \lambda x)^2} \tag{25}$$

A positive numerator implies the reassuring downward derivative with respect to age. It should be noted that Ξ is preceded by a minus sign—which we leave in place without including it in our variable—and that the denominator cannot be negative.

However, we will try to establish when the normal trend of life expectancy reverses, increasing with age. To this end we set up the inequality:

$$\Xi(x) < 0 \iff x \in \left[\frac{\omega-\sqrt{\omega(\lambda-1)(\omega-\lambda K)}}{\lambda}, \frac{\omega+\sqrt{\omega(\lambda-1)(\omega-\lambda K)}}{\lambda}\right] \tag{26}$$

The right-hand value of the interval for x would be greater than K^{Max} (formally, we could write $x^R > K^{Max}$, where x^R stands for "right", just as x^L stands for "left") and thus becomes irrelevant; let's rewrite then, also using survival probabilities

$$x > x^L = \frac{\omega-\sqrt{\omega(\lambda-1)(\omega-\lambda K)}}{\lambda} = \left[1-\sqrt{(\lambda-1)p(0,K)}\right]K^{Max} \to \frac{\partial e_1(x)}{\partial x} > 0 \tag{27}$$

Now we evaluate Ξ in the various possible cases, divided into three parts ordered by the values of λ, within which we will distinguish a few subcases.

3.1 La dolce vita: $\lambda \in [0,1)$

From equation (25) it is clear that

$$\begin{aligned}\Xi &:= \lambda^2 x^2 - 2\lambda\omega x + [(2-\lambda)\omega - \lambda(1-\lambda)K]\omega \\ &= (\omega-\lambda x)^2 + (1-\lambda)(\omega-\lambda K)\omega\end{aligned} \tag{28}$$

This reformulation makes it clear that Ξ will always be positive here; consequently, the derivative (25) will be trivially negative, as expected when the instantaneous mortality is modest. Nevertheless, we will provide a few brief considerations for the notable sub-case $\lambda = 0$, $\Xi = 2\omega^2$; the (25), in its entirety, is instead −1: here, the life expectancy decreases by an amount equivalent to the time elapsed, without partial recoveries for the avoided mortality, which is zero.

3.2. The right path: $\lambda = 1$

We have here a classical Moivre, where the radical contained in (26) becomes zero: the solution would then be $x^{R/L} = K^{Max}$; however, it is better to directly calculate it: by substituting λ with 1 in (25), we simply obtain -1/2.

3.3. Life is hard (and short): $\lambda > 1$

Here things get more complicated (and interesting), because Ξ can be negative; since the term x^L in (27) now becomes decisive, it is important to study its main features. Since the maximum value of *x* here is *K* (recall that we are in the first segment of the function), we can try to see if x^L can be lower[4] thus allowing for our eagerly sought inversion:

$$x^L_{\lambda>1} < K \rightarrow \omega - \lambda K < \sqrt{\omega(\lambda-1)(\omega-\lambda K)} \rightarrow \omega - \lambda K < \omega(\lambda-1) \quad (29)$$

By solving (29) for *K*, we obtain:

$$x^L_{\lambda>1} < K \Leftrightarrow K > (2-\lambda)\frac{\omega}{\lambda} = (2-\lambda)K^{Max} \quad (30)$$

Alternatively, we can also proceed by considering λ:

$$x^L_{\lambda>1} < K \Leftrightarrow \lambda > \frac{2\omega}{K+\omega} \in [1,2] \quad (31)$$

In both formulations, it is evident that the inversion can occur when the slope is steeper than that in the linear case; moreover, since $K \leq \omega$ it appears with certainty for $\lambda > 2$.

Once it is established when the inversion can (or must) occur, it remains to see if we start from birth in negative territory; this is also a matter of great interest because—keeping in mind that, as noted when discussing (26), x^R cannot interfere—in this case, the inversion would occur throughout the entire first segment of the function. Note that this is a sufficient condition, whereas the previous one was necessary.

From (27), we can derive:

$$x^L_{\lambda>1} < 0 \rightarrow \omega < (\lambda-1)(\omega-\lambda K) \rightarrow (\lambda-2)\omega > (\lambda-1)\lambda K \quad (32)$$

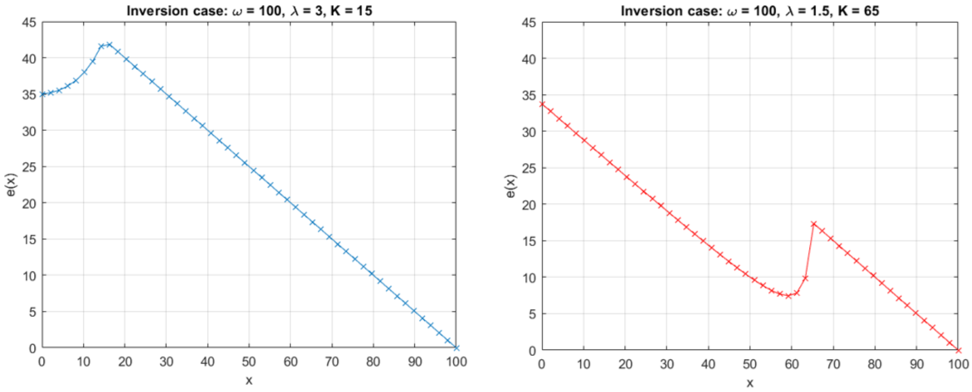
Then we obtain:

$$x^L_{\lambda>1} < 0 \rightarrow K < \frac{\lambda-2}{\lambda-1} K^{Max} \quad (33)$$

[4] Note that from (27), we already know that it cannot be greater than K^{Max} and certainly not greater than ω.

Hence, the inversion at all ages (of the first segment, obviously) is possible only if $\lambda > 2$.

Figure 2 – *Inversion of the function e(x) with* $\lambda > 2$ *(left) and* $\lambda \in (1,2)$ *(right).*



4. Conclusions

The present work investigates some theoretical and mathematical properties of the function $S(x)$, highlighting key aspects that are seldom addressed in existing models. A particularly intriguing result is the observed growth of $e(x)$ with age, which appears to be linked to the pronounced angularity of the piecewise linear structure of $S(x)$. Let us add also the importance of $\lambda = 2$ as threshold (this holds true also for $\lambda = 1$, but this was expected); notable also how some final outcome can be seen as a combination of the two original linear functions (see Appendix).

To the best of our knowledge, these behaviours are scarcely explored in current frameworks, suggesting potential avenues for further investigation and possible applications.

Looking ahead, an interesting direction for future research involves the generalization to quadratic or polynomial forms of $S(x)$. Specifically, a rigorous theoretical analysis could aim to determine whether polynomial curves $S(x)$ inherently preclude the growth of life expectancy $e(x)$ with age—or conversely, whether such growth can occur under specific conditions. Addressing this question would deepen our understanding of the interplay between $S(x)$ structure and $e(x)$.

Appendix

Frankenstein function

We can consider the piecewise function as the grafting of two linear functions, one for each segment: from the first one, *A*, we obtain the first line, from 0 to *K*; from the second function, *B*, we obtain the segment from *K* to ω.

A has a maximum age $\widehat{\omega}^A$, different from the global ω: if $S_1(x)$, shown in (7), extended beyond *K*, it would nullify for:

$$S(0)\left(1-\frac{\lambda}{\omega}x\right)=0 \rightarrow \ x \coloneqq \widehat{\omega}^A=\frac{\omega}{\lambda} \tag{34}$$

From which we can write it as a classic De Moivre with a new longevity:

$$S^A(x)=S(0)\left(1-\frac{x}{\widehat{\omega}^A}\right) \tag{35}$$

In the case of *B* we are looking for that initial contingent $S^B(0)$ that can adapt the second segment of the original piecewise function. We have:

$$S^B(x) \coloneqq S_2(x) \ \rightarrow \ S^B(0)=S(0)\left(\frac{\omega-\lambda K}{\omega-K}\right) \tag{36}$$

Hence we get a De Moivre, with different initial contingent. As we see by rewriting the second line of (7):

$$S^B(x)=S(0)\left(\frac{\omega-\lambda K}{\omega-K}\right)\left(1-\frac{x}{\omega}\right) \tag{37}$$

Moreover, it can be proved, but we omit it due to space limitations, that $e(0)$ is equal to the sum of $e^A(0)$ and $e^B(0)$, weighted with $c_0^A=\frac{\lambda K}{\omega}=\frac{\lambda}{\lambda^{Max}}=\frac{K}{K^{Max}}$ and $c_0^B=1-c_0^A$.More in-depth developments of this idea are currently under investigation.

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