BOOTSTRAP DOUBLE FREQUENCY DICKEY FULLER TEST FOR UNIT ROOTS

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Abstract. In this paper we present some advances over the Double Frequency Dickey Fuller test for unit root, recently proposed in literature to capture those situations where the time series might be affected by potential unknown structural breaks, asymmetrically located.
The Double Frequency Dickey Fuller is based on the idea the Fourier approach allows capturing the behavior of a deterministic function form even if it is not periodic, working better than dummy variables, independent of the breaks are instantaneous or smooth and avoiding the problem of selecting the dates and the form of the breaks. For the Double Frequency Dickey Fuller test, it has been developed the asymptotic theory and, via simulations, its finite sample properties have been shown with respect to a variety of processes.
To the best of our knowledge, however, no results have yet been presented to combine this test with using bootstrap methods, in order to possibly improve the finite sample performance. To address this issue, we propose a bootstrap test based on the sieve bootstrap for unit root and intend to conduct an extensive Monte Carlo experiment to evaluate its finite sample performance.

1. Introduction

Traditional unit root tests suffer from a loss of power when structural breaks in the data-generating process are ignored (Perron, 1989). By highlighting this issue, Perron (1989) initiated a new line of research focused on developing unit root tests that are robust to structural breaks or outliers in the data. This presents a significant challenge for applied economists, as the number, duration, and nature of structural breaks are often unknown. Furthermore, efforts to detect the number or location of these breaks may introduce pre-testing bias. Adding to the complexity, structural breaks occurring in a given year may not have an immediate or fully observable impact.

Early studies (Perron, 1989; Zivot and Andrews, 2002; Lee and Strazicich, 2003) addressed this issue by incorporating dummy variables to represent structural breaks. However, this approach can introduce an excessive number of nuisance parameters, limiting its practical usefulness. This limitation has motivated the development of alternative unit root and stationarity tests. Becker *et al.* (2006), for example, proposed tests that model structural breaks of unknown form as smooth transitions using Fourier transforms. Building on earlier work by Gallant (1981), they showed that Fourier approximations can effectively capture the behavior of unknown functions, even if the function is not periodic. The Fourier-based approach only requires specifying the appropriate frequency in the estimation process, thus reducing the number of parameters that must be estimated. As a result, these tests demonstrate good size and power properties, regardless of the timing or shape of the break, and outperform dummy-based methods. In line with this framework, recent contributions have extended the original approach by Becker *et al*. (2006). Notably, Enders and Lee (2012) proposed a testing framework that incorporates the Fourier transform while avoiding the need to pre-specify the timing, number, or form of the breaks.

Omay (2015) advanced this line of research by combining the methodologies of Becker *et al.* (2006) and Enders and Lee (2012), introducing the use of fractional frequencies to enhance model fit. More recently, Cai and Omay (2022) proposed a Double Fourier frequency test capable of capturing asymmetrically located structural break, as in the case of the nominal crude oil price or relative commodity prices (see Cai and Omay 2022 for further details).

In this paper we move from Cai and Omay's (2022) test, a Dickey-Fuller type test for which the authors derived the asymptotic distribution (tabulated for some specific sample sizes). To the best of our knowledge, no attempt has been made to propose a bootstrap version of the Double Fourier Dickey-Fuller test and, considering the efficacy of bootstrap methods, in this work we intend to address this issue.

Bootstrap methods (Efron, 1979) are a computer intensive approach, based on resampling, to statistical inference issues without any statistical assumption on the underlying data generating process. The bootstrap is often adopted because it has much better performance than the conventional approaches (first order asymptotics) and it provides empirical and efficient inference for complicated problems. However, bootstrap methods cannot be equally applicable to all random processes.

In general, for data that are not independent identically distributed (iid), modifications of the original bootstrap set-up have been put forward in order to be able to resample without ignoring the dependence structure. Traditional bootstrap methods for time series can be categorized into parametric, semiparametric and non parametric approaches. Parametric methods resample residuals from a correctly specified model with parameters estimated from the observed data (Andrews *et al*., 2006). These methods assume that residuals are independent and identically distributed. Semi and non parametric methods, such as the Sieve Bootstrap (Buhlmann, 1997) and the Block Bootstrap (Kunsch, 1989), resample approximately independent quantities, like residuals or blocks of data. While these methods are effective for short-memory series, they struggle with unit root processes, where the strong persistence of dependence violates the independence assumption. To address this, alternative strategies, such as pre-filtering the data (Poskitt *et al.*, 2015) or resampling in the frequency domain (Franke and Hardle, 1992), have been developed.

In this paper we propose a Sieve Bootstrap Double Frequency Dickey Fuller (SB-DFDF) test for unit root, by combining the use of the Sieve Bootstrap with the Double Frequency Dickey Fuller test in case the time series might be affected by potential unknown structural breaks, asymmetrically located. In the second section, we will present the Double Frequency Dickey Fuller test. In the third section, we present our algorithm for the Sieve Bootstrap Double Frequency Dickey Fuller test. In the fourth section, we will present our Monte Carlo experiment and some conclusions.

2. Double Frequency Dickey Fuller Test

The modification of the Dickey-Fuller test (Dickey and Fuller, 1979, 1981) to account for a deterministic function d_t moves from the following AR(1) process with a deterministic trend

$$y_t = d_t + \rho y_{t-1} + \varepsilon_t \qquad t = 1, \dots, T \tag{1}$$

where the stationary term ε_t has variance σ^2, d_t is a deterministic function. If d_t is known, model (1) can be directly estimated and, in turn, the unit root hypothesis $H_0\!: \rho = 1$ can be tested. When d_t is unknown testing for unit root is problematic given the risk of misspecification of d_t. The idea underlying the Fourier expansion-based Dickey-Fuller test is that it is often possible to approximate d_t using the Fourier expansions, as in Enders and Lee (2012):

$$d_t = \alpha_0 + \sum_{k=1}^{n} \alpha_k \left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{n} \beta_k \left(\frac{2\pi kt}{T}\right) \qquad n \leq T/2 \tag{2}$$

where n represents the number of cumulative frequencies included in the approximations and k represents a particular frequency. It is interesting to observe that in the absence of a nonlinear trend, all values $\alpha_k = \beta_k = 0$, so that the usual Dickey-Fuller specification appears. Usually the number of frequencies n should be kept small to avoid overfitting; in particular, in the original idea of Enders and Lee (2012), the Fourier approximation is adopted for a single frequency (n=1) as follows

$$d_t = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi k t}{T}\right) + \beta \cos\left(\frac{2\pi k t}{T}\right) \quad (3)$$

that includes, via the first term in the sum where $i = 0{,}1$, both the intercept and the trend plus intercept versions and it also approximates, via the sinusoidal waves the smooth breaks. In expression (3), k is the frequency to be determined over a pre-given interval. However, as pointed by Omay (2015) the breaks caused by sudden geo political events and financial crisis are stochastically distributed and asymmetrically located. Following this logic, Cai and Omay (2022) relax the assumption that the frequency is identical and propose a more general set up where:

$$d_t^{Dfr} = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi k_s t}{T}\right) + \beta \cos\left(\frac{2\pi k_c t}{T}\right) \quad (4)$$

and within the framework of a Dickey-Fuller unit root test, the model with optimal frequencies k_s and k_c is:

$$y_t = \sum_{i=0}^{1} c_i t^i + \alpha \sin\left(\frac{2\pi k_s t}{T}\right) + \beta \cos\left(\frac{2\pi k_c t}{T}\right) + \rho y_{t-1} + \varepsilon_t, \quad (5)$$

and the Double Frequency Dickey Fuller (DFDF) test statistic for $H_0\colon \rho = 1$ is:

$$\tau^{Dfr} = \frac{T(\hat{\rho}-1)}{\sqrt{T^2 \delta_{\hat{\rho}}^2}} \quad (6)$$

where $\hat{\rho}$ and $\delta_{\hat{\rho}}^2$ are OLS estimators of ρ and standard errors. The asymptotic distribution of the test statistic τ^{Dfr} only depends on the frequencies k_s and k_c and the critical values are tabulated (Cai and Omay, 2022, table 1).

If a nonlinear trend is not actually present in the data, a standard unit root test, such as Dickey-Fuller (DF) or Augmented Dickey Fuller (ADF, Said and Dickey, 1984) is more powerful and there is no need of Fourier terms. So, before adopting the DFDF test with a predetermined frequency pair (k_s, k_c), it recommended to test $H_0\colon linearity$ versus $H_1\colon non\ linearity$ via an adjusted F test:

$$F^{Dfr}(k_s, k_c) = \frac{\frac{SSR_0 - SSR_1(k_s,k_c)}{2}}{\frac{SSR_1(k_s,k_c)}{T-q}} \quad (7)$$

SSR_0 and $SSR_1(k_s, k_c)$ represent sum of squared residuals without and with Fourier components, q is the number of regressors. If H_0 is rejected, a functional form with Fourier components is suggested.

Selecting the double frequency is done with a grid search to find the optimal pair (k_s^*, k_c^*), through the minimization of the SSR. This leads to the modified F test:

$$F^{Dfr}(k_s^*, k_c^*) = max_{(k_s,k_c)}F^{Dfr}(k_s, k_c)$$

where $(k_s^*, k_c^*) = argmaxF^{Dfr}(k_s, k_c)$. Minimizing SSR is equivalent to maximizing the F^{Dfr} test statistic under the condition of maximum frequency k_{max} and a searching precision of Δk (critical values tabulated).

3. Sieve Bootstrap Double Frequency test for Unit Root

Instead of resorting on tabulated critical values as in the original version of the test by Cai and Omay (2022), here we propose to derive the empirical distribution of the statistics via Sieve Bootstrap. The Sieve Bootstrap (Buhlmann, 1997) approximates a time series with an autoregressive process of increasing order p, from which residuals are resampled. This method is semi-parametric, relying on a flexible model approximation rather than a fully specified data generating process (DGP).

Originally designed for short-memory processes, its applicability has been extended to fractionally integrated and non-invertible processes by Poskitt (2008). Moreover, when strong long-range dependence dominates the data, such as unit root, even advanced resampling techniques may fail to capture the underlying distributional properties. Therefore, Poskitt *et al*. (2015) proposed Pre-Filtered Sieve Bootstrap whose algorithm first takes the first difference of the time series and then applies the Sieve Bootstrap to the filtered (short-memory) series. The final bootstrap sample is obtained by inverting the first difference transformation.

Considering that here, we work within the unit root set-up, our bootstrap version of the DFDF test would be effectively a Prefiltered-Sieve Bootstrap Double Frequency Dickey Fuller test and below it is presented the overall algorithm.

Step	Action	Formula/Description
Start	**Initialize**	Given a time series y_t, $t = 1,...,T$ and a statistic S.
1	**Achieve Stationarity**	Take the first difference: $x_t = \Delta y_t$
2	**Model & Get Residuals**	Model x_t, with AR(p) (p chosen with AIC) and get the residuals ε_t
3	**Resample Residuals**	**Resample** the residuals ε_t, with replacement to obtain the bootstrap residuals ε_t^*
4	**Generate Bootstrap Series (Stationary)**	Obtain the bootstrap time series x_t^* using the estimated AR(p) coefficients (α_i) and the resampled residuals: $x_t^* = \alpha_0 + \alpha_1 x_{t-1}^* + \cdots + \alpha_p x_{t-p}^*$
5	**Reconstruct Bootstrap Series (Original Scale)**	Reconstruct the final bootstrap sample y_t^* by "undifferencing" x_t^*: $y_t^* = \Delta^{-1} x_t^*$
6	**Compute Statistic**	Compute the desired statistic S on the bootstrap sample: $S_{b^*}=S(y_t^*)$.
Loop	**Repeat (B Times)**	**Repeat Steps 1-6 B times** to generate B bootstrap samples y_t^* and calculate B values of the statistic S_{b^*}
7	**Estimate Distribution**	Approximate the distribution of the statistic S using the empirical bootstrap distribution FS∗: $FS^* = \frac{1}{B}\sum^B I(S_b^* \leq s)$ (8)
End	**Result**	Bootstrap Distribution of Statistic S

4. Monte Carlo experiment

In this section we present our Monte Carlo experiment to ascertain the finite sample properties of the Sieve Bootstrap Double Frequency Dickey Fuller test (SB-DFDF), in comparison with DFDF test by Cai and Omay (2022). The test is applied using the optimal k_s^*, k_c^*, according to the grid search procedure presented in the second section and no preliminary F test is carried out, meaning that both tests' performance is evaluated assuming that the presence of breaks in the DGP is known. The sample size is T=50,150, 300, the number of simulations is 2000. For the implementation of the SB-DFDF test, the number of bootstrap replications is 500 and the nominal size is 0.05. Simulations have been done with R (Team, 2013).

We consider the following data generating processes (DGPs), following Enders and Lee (2012), where for each of them $\rho = 1, 09, 08, 07$ (the values of the other characterizing parameters are displayed in the result tables).

- AR(1) with smooth shifts with constant level (DGP 1a)

$$y_t = \alpha_0 + \sum_{k=1}^{n} \alpha_k \left(\frac{2\pi k_s t}{T}\right) + \sum_{k=1}^{n} \beta_k \left(\frac{2\pi k_c t}{T}\right) + \rho y_{t-1} + \varepsilon_t \quad t = 1, \dots, T$$

- AR(1) with smooth shifts with linear trend (DGP 1b)

$$y_t = \alpha_0 + c_1 t + \sum_{k=1}^{n} \alpha_k \left(\frac{2\pi k_s t}{T}\right) + \sum_{k=1}^{n} \beta_k \left(\frac{2\pi k_c t}{T}\right) + \rho y_{t-1} + \varepsilon_t \quad t = 1, \dots, T$$

- Sharp shifts (DGP 2)

$$y_t = T_t + e_t \quad t = 1, \dots, T$$

where $e_t = \rho e_{t-1} + \varepsilon_t$ and $T_t = \begin{cases} b & if\ t \leq aT \\ 0 & otherwise \end{cases}$

- LStar smooth shifts (DGP 3a)

$$y_t = \frac{1}{1 + e^{-\varphi(t-\theta T)}} + e_t \quad t = 1, \dots, T$$

where $e_t = \rho e_{t-1} + \varepsilon_t$

- Estar smooth shifts (DGP 3b)

$$y_t = d\left(1 - e^{-\varphi(t-\theta T)^2}\right) + e_t \quad t = 1, \dots, T$$

where $e_t = \rho e_{t-1} + \varepsilon_t$

The performance of the tests is expressed in terms of percentages of rejections. This percentage represents the empirical size under H_0, that is when in the DPGs $\rho = 1$, and it represents empirical power under H_1, that is for the other values of ρ. The results are presented in the set of tables below (tables 1-6).

As can be seen from the tables, the SB-DFDF test shows good performance in terms of empirical size (always around 0.05), being very similar to the DFDF test in this respect. Regarding empirical power, there are some differences between the two tests, indicating a systematically better performance of the SB-DFDF test compared to the DFDF test.

Analyzing the individual DGPs, we can observe that in the cases of DGP1a and DGP1b (Tables 1 and 2), the DGPs that are most similar to the logic of the DF set-up, both tests show very good power, which, as expected, only decreases when the autoregressive parameter ρ approaches 1.

Moving towards the DGPs that are more distant in logic from DF set-up, and starting from DGP2, we can appreciate a drop in the power of both tests in the presence of sharp shifts, which does not seem to be affected by the magnitude of the break or its position in the beginning or end of the series. It should be noted, however, that at sample size 300, the power already reaches excellent levels. Again, proximity of the autoregressive parameter to 1 reduces power.

As for DGP3a and DGP3b, Tables 5-6 show the performance in case of L-star smooth breaks, DGP3a (due to space constraints, the E-star results are not shown, but they are very similar to those of the L-star and available upon request from the authors). As in the previous case, the power performance is somewhat lower compared to the DGP1 cases, but still better than in the DGP2 case. Again, we observe the usual reduction in power as the autoregressive parameter increases from 0.7 to 0.9.

We also note, particularly for DGP2, DGP3a (and DGP3b), that the performance of the test demonstrates robustness regardless of whether the test is implemented with or without a linear trend in the specification.

Table 1 –*DGP 1a: AR(1) with constant level, percentage of rejection of null hypothesis (nominal size=5%, critical values for DFDF from Cai and Omay (2022) tables).*

ρ	k_s^*, k_c^*	T=50		T=150		T=300	
		DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	1,1	0.494	0.526	0.994	0.888	1	1
	1,2	0.434	0.514	0.844	0.898	1	1
	2,1	0.406	0.586	0.852	0.876	1	1
	2,2	0.416	0.534	0.904	0.874	1	1
0.8	1,1	0.328	0.444	0.800	0.888	1	1
	1,2	0.356	0.480	0.732	0.856	1	1
	2,1	0.322	0.522	0.878	0.896	1	1
	2,2	0.308	0.518	0.788	0.799	1	1
1	1,1	0.051	0.052	0.048	0.050	0.047	0.049
	1,2	0.052	0.052	0.052	0.051	0.049	0.050
	2,1	0.052	0.052	0.051	0.052	0.050	0.049
	2,2	0.052	0.053	0.054	0.049	0.049	0.050

Overall, we can conclude that, given our empirical results, it seems that the SB-DFDF test manages to improve the performance of the original DFDF, thus leading to powerful tool that is able to handle potentially asymmetrically located structural breaks when testing for nonstationarity, avoiding the use of nonstandard asymptotic distributions. Moreover, the test SB-DFDF test has reasonable power properties also in the presence of breaks that are not necessarily generated from the trigonomic components in the DGP.

We believe these results are interestingly promising and future research lines include the use of alternative bootstrap methods that can be even more suitable for the peculiarity of the test and data, for example frequency based bootstrap methods.

Table 2 – *DGP 1b: AR(1) with linear trend, percentage of rejection of null hypothesis (nominal size=5% critical values for DFDF from Cai and Omay (2022) tables).*

ρ	k_s^*, k_c^*	T=50		T=150		T=300	
		DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	1,1	0.448	0.758	0.894	0.888	0.980	0.999
	1,2	0.416	0.449	0.840	0.898	1	1
	2,1	0.456	0.448	0.798	0.876	0.999	0.998
	2,2	0.434	0.505	0.794	0.874	1	0.999
0.8	1,1	0.274	0.335	0.744	0.888	0.999	0.999
	1,2	0.252	0.323	0.750	0.856	0.970	0.999
	2,1	0.265	0.369	0.714	0.896	0.999	0.999
	2,2	0.255	0.354	0.746	0.799	1	1
1	1,1	0.051	0.052	0.045	0.050	0.045	0.045
	1,2	0.053	0.052	0.051	0.052	0.049	0.049
	2,1	0.050	0.050	0.051	0.053	0.054	0.049
	2,2	0.052	0.051	0.052	0.050	0.049	0.048

Table 3 –*DGP 2: Sharp shifts percentage of rejection of null hypothesis, test implemented without linear trend, nominal size=5%, critical values for DFDF from Cai and Omay (2022) tables.*

ρ	b	a	T=50		T=150		T=300	
			DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	1	0.2	0.258	0.342	0.737	0.675	0.985	0.989
		0.8	0.301	0.374	0.662	0.726	0.995	0.985
	5	0.2	0.205	0.180	0.384	0.473	0.821	0.866
		0.8	0.189	0.177	0.422	0.361	0.690	0.650
0.9	1	0.2	0.157	0.154	0.223	0.258	0.225	0.327
		0.8	0.149	0.136	0.189	0.210	0.299	0.296
	5	0.2	0.156	0.154	0.223	0.268	0.215	0.337
		0.8	0.119	0.125	0.170	0.210	0.209	0.204
1	1	0.2	0.053	0.054	0.052	0.050	0.049	0.050
		0.8	0.052	0.052	0.051	0.051	0.050	0.050
	5	0.2	0.052	0.052	0.052	0.052	0.051	0.051
		0.8	0.052	0.051	0.051	0.049	0.047	0.050

Table 4 –*DGP 2: Sharp shifts percentage of rejection of null hypothesis, test implemented with linear trend, nominal size=5%, critical values for DFDF from Cai and Omay (2022) tables.*

ρ	b	a	T=50		T=150		T=300	
			DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	1	0.2	0.247	0.347	0.604	0.702	0.927	0.931
		0.8	0.298	0.334	0.625	0.684	0.804	0.804
	5	0.2	0.184	0.180	0.363	0.421	0.661	0.662
		0.8	0.188	0.185	0.335	0.345	0.625	0.631
0.9	1	0.2	0.157	0.160	0.210	0.221	0.333	0.384
		0.8	0.156	0.159	0.198	0.220	0.325	0.365
	5	0.2	0.155	0.159	0.190	0.224	0.302	0.378
		0.8	0.156	0.159	0.188	0.220	0.300	0.375
1	1	0.2	0.050	0.051	0.052	0.052	0.049	0.045
		0.8	0.051	0.050	0.049	0.049	0.050	0.048
	5	0.2	0.050	0.052	0.048	0.048	0.051	0.047
		0.8	0.052	0.051	0.052	0.052	0.047	0.045

Table 5 –*DGP 3a: L-star smooth shifts percentage of rejection of null hypothesis, test implemented without linear trend, nominal size=5%, critical values for DFDF from Cai and Omay (2022) tables.*

ρ	a	T=50		T=150		T=300	
		DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	0.25	0.556	0.668	0.745	0.801	0.944	0.984
	0.5	0.498	0.601	0.628	0.703	0.976	0.996
	0.75	0.506	0.628	0.706	0.805	0.928	0.998
0.9	0.25	0.122	0.121	0.140	0.152	0.203	0.190
	0.5	0.122	0.119	0.153	0.144	0.260	0.240
	0.75	0.136	0.124	0.150	0.176	0.214	0.235
1	0.25	0.051	0.050	0.049	0.052	0.048	0.048
	0.5	0.052	0.053	0.054	0.052	0.051	0.045
	0.75	0.052	0.052	0.052	0.052	0.050	0.051

Table 6 –*DGP 3a: L-star smooth shifts percentage of rejection of null hypothesis, test implemented with linear trend, nominal size=5%, critical values for DFDF from Cai and Omay (2022) tables.*

ρ	a	T=50		T=150		T=300	
		DFDF	SB DFDF	DFDF	SB DFDF	DFDF	SB DFDF
0.7	0.25	0.504	0.618	0.725	0.799	0.924	0.982
	0.5	0.490	0.579	0.632	0.713	0.976	0.996
	0.75	0.516	0.638	0.721	0.845	0.938	0.990
0.9	0.25	0.131	0.121	0.140	0.152	0.203	0.190
	0.5	0.122	0.119	0.153	0.149	0.259	0.238
	0.75	0.139	0.124	0.150	0.176	0.214	0.235
1	0.25	0.052	0.051	0.050	0.052	0.051	0.051
	0.5	0.052	0.053	0.051	0.052	0.049	0.051
	0.75	0.052	0.051	0.052	0.052	0.050	0.049

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